FOCUSING OF ELECTROMAGNETIC PLANE WAVE INTO UNIAXIAL CRYSTAL BY A THREE DIMENSIONAL PLANO CONVEX LENS

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Abstract—A three dimensional plano-convex lens which is placed at a certain distance from a plane uniaxial interface has been considered. High frequency fields refracted by the geometry are derived. The treatment is based on Maslov's method. The method combines the simplicity of asymptotic ray theory and generality of the transform method to remedy the problem of geometrical optics around the caustic point of a focusing system. Field patterns are obtained which includes the observation points around the caustic region. The results are found in good agreement with obtained using Huygens-Kirchhoff Principle.

1. INTRODUCTION

Study of focusing of electromagnetic waves into dielectric media is a subject of considerable current interest due to its various important and useful applications, e.g., hyperthermia, microscopy, and optical data storage. Investigations of field in focal space of different focusing systems have been carried out [1–9]. We have analyzed geometry of a focusing system by employing the Maslov's method [10]. According to Maslov's method, the field expression near the caustic can be constructed by using the information in geometrical optics field. These information are used to find unknowns in the assumed spectrum representation of the field. This means that Maslov's method utilizes both simplicity of asymptotic ray theory and generality of transform method. That is singularity in one domain does not overlap in transform domain. Maslov's method has been reviewed with a view to applications by Ziolkowski and Deschamps [11] and Kravtsov [12]. This method has been successfully applied to predict the field in the focal region of parabolic reflector, phase transformer, cylindrical reflector,

spherical reflector antenna, dielectric spherical lens, spherical dielectric interface, plano-convex antenna, Cassegrain system, Gregorian system, Wood lens, inhomogeneous slab, cylindrical reflector placed in chiral medium, cylindrical reflector coated with chiral layer, a PEMC reflector and a hyperbolic lens by different researchers who worked with Hongo and/or Naqvi [13–30]. Study fields in uniaxial, anisotropic and bi-anisotropic material has been discussed by various authors [31–41].

Though the Huygens-Kirchhoff Principle and Maslov's method are of comparable accuracy, Maslov's method has some distinct advantages for specific problems. When Kirchhoff's integral is applied to the these problem, we must perform a double integration even though the geometry has cylindrical symmetry because we cannot use the Fresnel approximation for the kernel. Maslov's method has advantage over Huygens-Kirchhoff Principle when we deal with the field in focal region in the special case of axially symmetrical configurations like has been treated in the present problem. It is faster to compute the data from final field expression than other methods.

In present discussion, we have extended our previous work [23] to three dimensional case. Here the field refracted by a focusing geometry which contains a three dimensional plano-convex lens at a certain distance from plane uniaxial interface has been considered. For the special case in which the incident field is a transverse magnetic (TM) plane wave polarized in the xz plane, which is the plane of incidence, considerable simplifications occur, especially if we also let the optical axis in the crystal to be in the plane of incidence. In this case the ordinary and extraordinary waves inside anisotropic medium are TE and TM waves, respectively. We have studied the special case of focusing of extraordinary plane waves into uniaxial crystal.

2. DERIVATION OF THE FIELD IN A PLANO CONVEX LENS

Consider the geometry as shown in Figure 1. It contains a 3d-plano convex lens placed apart from a uniaxial crystal interface. Front face of the lens is located at $z = d$ while rare face is located at $z = d_0$ and uniaxial crystal interface is at $z = \zeta_1$. Uniaxial crystal occupies half space $z \geq \zeta_1$. Electromagnetic plane wave polarized in x-direction and propagating in z-direction, is incident on a plano convex lens. After passing through the plano convex lens, ray is refracted through plane interface of uniaxial crystal. It is assumed that uniaxial crystal occupying the half space $z \geq \zeta_1$ has principle permittivities (ϵ^o, ϵ^e) and permeability μ_2. Half space $z < \zeta_1$ has constitutive parameters
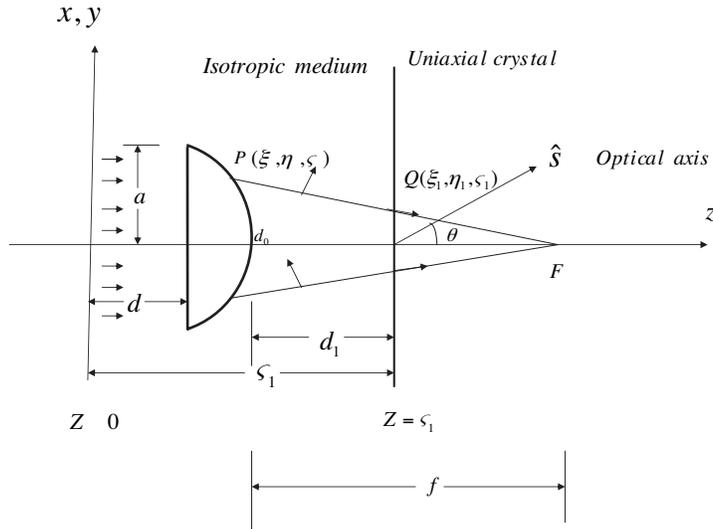
Figure 1. Geometry for focusing of 3D electromagnetic waves into uniaxial crystal.

(ϵ_1, μ_1). The equation describing the surface of the plano convex lens is given by

$$\zeta = g(\rho) = \frac{n}{n+1}f - \frac{1}{\sqrt{n^2-1}}\sqrt{\rho^2 + \frac{n-1}{n+1}f^2}, \quad \rho = \sqrt{\xi^2 + \eta^2} \quad (1)$$

where (ξ, η, ζ) are the Cartesian coordinates of the point on the plano convex lens. n is the refractive index of plano-convex lens and f is the focal length.

Incident wave is given by

$$E_x^i = \exp(-jkz) \tag{2}$$

where $k = \frac{\omega}{c}\sqrt{\epsilon_1\mu_1}$. Our interest is to determine the transmitted field into the uniaxial crystal. First we obtain the field transmitted through the lens into isotropic medium. The Cartesian coordinates (x, y, z) of the ray after passing through the lens may be described using the solutions of Hamilton's equations. That is

$$x = \xi + q_x t, \qquad y = \eta + q_y t, \qquad z = \zeta + q_z t$$

where (q_x, q_y, q_z) are ray vector components and t is the parameter along the ray. Solutions of Hamilton's equations develop relation

between Cartesian coordinates (x, y, z) and ray coordinates (ξ, η, t). The Cartesian coordinates of the ray at front surface of the uniaxial crystal are given by

$$\xi_1 = \xi + q_x t_1, \qquad \eta_1 = \eta + q_y t_1, \qquad \zeta_1 = \zeta + q_z t_1 \qquad (3)$$

where $t_1 = \frac{\zeta_1 - \zeta}{q_z}$ is distance between the point $P(\xi, \eta, \zeta)$ on curved surface of convex lens and the point $Q(\xi_1, \eta_1, \zeta_1)$ on front face of uniaxial crystal. (ξ, η, ζ) are rectangular coordinates of initial point on the refracted ray or point on the curved surface of the lens. The Cartesian coordinates of the ray refracted into the uniaxial crystal are given by

$$\begin{aligned}
x &= \xi_1 + q_x^u t = \xi + q_x t_1 + q_x^u t, \\
y &= \eta_1 + q_y^u t = \eta + q_y t_1 + q_y^u t, \\
z &= \zeta_1 + q_z^u t = \zeta + q_z t_1 + q_z^u t
\end{aligned} \qquad (4)$$

where t is the parameter along the ray traveling in the uniaxial crystal. (ξ_1, η_1, ζ_1) are rectangular coordinates of initial point on the refracted ray or point on the front face of uniaxial crystal. It may be noted that (q_x, q_y, q_z) are components of ray before uniaxial crystal while (q_x^u, q_y^u, q_z^u) are components of ray in uniaxial crystal.

To find components of ray vector in region between lens and uniaxial crystal, we need expression for the normal to the curved face of the lens. Unit normal $\mathbf{N}$ of the curved surface is given by

$$\mathbf{N} = \sin \alpha \cos \beta \mathbf{i}_x + \sin \alpha \sin \beta \mathbf{i}_y + \cos \alpha \mathbf{i}_z \qquad (5)$$

where (α, β) are angular polar coordinates of the point (ξ, η, ζ) defined by

$$\begin{aligned}
\xi &= \rho \cos \beta \\
\eta &= \rho \sin \beta \\
\zeta &= g(\rho) \\
\rho &= \frac{(n-1)f \tan \alpha}{\sqrt{1 - (n^2 - 1) \tan^2 \alpha}} \\
\sin \alpha &= -\frac{g'(\rho)}{\sqrt{1 + (g'(\rho))^2}} \\
\cos \alpha &= \frac{1}{\sqrt{1 + (g'(\rho))^2}} \\
\tan \beta &= \frac{\eta}{\xi}
\end{aligned} \qquad (6)$$

In above equations prime represents derivative. The ray vector of the refracted ray by plano convex lens may be obtained using the relation $\mathbf{q} = n\mathbf{p}^i + \sqrt{1 - n^2 + n^2(\mathbf{p} \cdot \mathbf{N})^2}\mathbf{N} - n(\mathbf{p} \cdot \mathbf{N})\mathbf{N}$, which is derived from Snell's law with n is the refractive indexes of the lens. The ray vector of the ray refracted by the plano convex lens is given by

$$
\begin{aligned}
\mathbf{q} &= K(\alpha)\sin\alpha\cos\beta\mathbf{i}_x + K(\alpha)\sin\alpha\sin\beta\mathbf{i}_y + (n + K(\alpha)\cos\alpha)\mathbf{i}_z \\
&= q_x\mathbf{i}_x + q_y\mathbf{i}_y + q_z\mathbf{i}_z
\end{aligned}
\tag{7}
$$

where

$$
K(\alpha) = \sqrt{1 - n^2\sin^2\alpha} - n\cos\alpha
$$

Ray refracted by the lens hits uniaxial crystal interface. The electromagnetic field that is incident on the plane interface is TM field. Due to this incidence, TE as ordinary wave and TM as extraordinary wave are excited inside uniaxial crystal half space. It is assumed that there is no coupling between TE and TM waves. Ray vector of the refracted ray into uniaxial crystal may be obtained as [5–7]

$$
\mathbf{q}^{\text{et}} = q_x\mathbf{i}_x + q_y\mathbf{i}_y + q_z^e\mathbf{i}_z
\tag{8}
$$

In above equation (8), superscript et means extra-ordinary transmitted. x and y components are same as in the medium before uniaxial interface while z component may be written as

$$
\begin{aligned}
q_z^e &= A + \sqrt{B} \\
A &= -\frac{\chi K(\alpha)\sin\alpha\sin\theta\cos\theta}{1 + \chi\cos^2\theta} \\
B &= \frac{(q^e)^2 - (K(\alpha)\sin\alpha)^2}{1 + \chi\cos^2\theta} - \frac{A^2}{\chi\cos^2\theta}
\end{aligned}
$$

where χ is measure of anisotropy in the uniaxial crystal and is given by

$$
\chi = \frac{(q^e)^2}{(q^o)^2} - 1, \qquad q^e = \frac{\omega}{c}\sqrt{\epsilon^e\mu_2}, \qquad q^o = \frac{\omega}{c}\sqrt{\epsilon^o\mu_2}
$$

where θ is angle of optical axis with z-axis.

Geometrical-optics solution are derived as

$$
\mathbf{E}^r(x,y,z) = \mathbf{E}_T(\xi,\eta)\left[J(t)\right]^{-\frac{1}{2}}\exp\left[-jk\left(S_0(\xi,\eta) + t\right)\right]
\tag{9}
$$

where $J(t)$ is the Jacobian of coordinate transformation from ray coordinates (ξ, η, t) to rectangular coordinates (x, y, z) and has been derived in the appendix

$$
J(t) = \frac{D(t)}{D(0)} = \frac{1}{D(0)}\frac{\partial(x,y,z)}{\partial(\xi_1,\eta_1,t)} = \left(-P\frac{U_0}{E}t + 1\right)\left(\frac{Q_t(\alpha)}{\rho}t + 1\right)
$$

where

$$P = \frac{\left(\sqrt{n^2-1}\right)^{\frac{n-1}{n+1}} f^2}{[n^2\xi^2 + (n^2-1)^2 f^2][\xi^2 + \frac{n^2-1}{n+1} f^2]^{\frac{1}{2}}}$$

$$\rho = \frac{(n-1)f \tan\alpha}{\sqrt{1 - (n^2-1)\tan^2\alpha}}$$

$$U_0 = Q_t \frac{\partial q_z^e(\alpha)}{\partial\alpha} - q_z^e \frac{\partial Q_t(\alpha)}{\partial\alpha})$$

$$E = q_z^e + Q_t \tan\alpha$$

$$Q_t = K(\alpha)\sin\alpha$$

$$\frac{\partial q_z^e}{\partial\alpha} = G\left(1 - \frac{1}{\sqrt{B}}\frac{K(\alpha)\sin\alpha}{1 + \chi\cos^2\theta} + \frac{A}{\sqrt{B}\chi\cos^2\theta}\right)$$

$$G = -\frac{\chi K(\alpha)\sin\theta\cos\theta}{1 + \chi\cos^2\theta}\left(\frac{-n^2\sin\alpha\sin 2\alpha}{2\sqrt{1-n^2\sin^2\alpha}} + n^2\sin^2\alpha + K(\alpha)\cos\alpha\right)$$

$$\frac{\partial Q_t(\alpha)}{\partial\alpha} = D_t = \frac{(1 - 2n^2\sin^2\alpha)\cos\alpha}{\sqrt{1 - n^2\sin^2\alpha}} - n\cos 2\alpha$$

$\mathbf{E}_T$ is the amplitude of the refracted ray at the refraction point. Initial phase S_0 on the surface of the lens and t are defined as

$$S_0 = n\zeta + t_1, \qquad t = \sqrt{(x-\xi_1)^2 + (y-\eta_1)^2 + (z-\zeta_1)^2} \qquad (10)$$

It is readily seen that the GO expression of the refracted ray becomes infinity at the point F as is expected.

3. DERIVATION OF THE EXPRESSION VALID IN CAUSTIC

According to Maslov's method, the three-dimensional expression for the field that is valid near the caustic is given by [10–12]

$$\mathbf{E}^r(\mathbf{r})\sqrt{\frac{k}{j2\pi}}\int_{-\infty}^{\infty}\mathbf{E}_T(\xi,\eta)\left[\frac{D(t)}{D(0)}\frac{\partial(q_x, q_y)}{\partial(x, y)}\right]^{-\frac{1}{2}}$$

$$\exp\left\{-jk\left[S_0 + t - x(q_x, q_y, z)q_x - y(q_x, q_y, z)q_y + q_x x + q_y y\right]\right\}dq_x dq_y$$

$$(11)$$

Equation (11) is derived by applying the stationary phase method to the conventional Fourier-transform representation for $\mathbf{E}^r(\mathbf{r})$ and comparing it with the geometrical optics field given in Eq. (9). Thus

the integrand of the inverse Fourier transform of the wave function is derived through the information of the GO solution. In (11), $x(q_x, z)$ means that the coordinate x should be expressed in terms of mixed coordinates (q_x, z) by using (6). The same is true for t and it is given by $t = \frac{z-\zeta}{q_z^e}$. The value of $\left[J(t)\frac{\partial(q_x,q_y)}{\partial(x,y)}\right]^{-\frac{1}{2}}$ is given by

$$
\begin{aligned}
\left[J(t)\frac{\partial(q_x, q_y)}{\partial(x, y)}\right]^{-\frac{1}{2}} &= \left[\frac{1}{D(0)}\left(\frac{\partial(q_x)}{\partial\xi}\frac{\partial(q_y)}{\partial\eta} - \frac{\partial(q_y)}{\partial\xi}\frac{\partial(q_x)}{\partial\eta}\right)\frac{\partial z}{\partial t}\right]^{-\frac{1}{2}} \\
&= \left[\frac{-PD_tQ_tq_z^e}{\rho E}\right]^{-\frac{1}{2}}
\end{aligned}
\tag{12}
$$

The phase function is given by

$$
\begin{aligned}
S &= S_0 + t + t_1 - x(q_x, q_y, z)q_x - y(q_x, q_y, z)q_y + q_x x + q_y y \\
&= S_0 + t_1 + q_x(x - \xi_1) + q_y(y - \eta_1) + q_z^e(z - \zeta_1) \\
&= S_0 + t_1 + q_x(x - \xi - q_x t_1) + q_y(y - \eta - q_y t_1) + q_z^e(z - \zeta_1) \\
&= S_0 + (\zeta_1 - \zeta)q_z + q_x(x - \xi) + q_y(y - \eta) + q_z^e(z - \zeta_1) \\
&= S_0 + (\zeta_1 - \zeta)q_z - \rho K(\alpha)\sin\alpha + Kr\sin\alpha\sin\theta_0\cos(\phi_0 - \beta) \\
&\quad + (z - \zeta_1)q_z^e
\end{aligned}
\tag{13}
$$

where

$$
\begin{aligned}
x &= r\sin\theta_0\cos\phi_0 \\
y &= r\sin\theta_0\sin\phi_0 \\
z &= r\cos\theta_0.
\end{aligned}
$$

Transforming the integration variables (q_x, q_y) into (α, β) that is,

$$
d(q_x)d(q_y) = \left[\frac{(1 - 2n^2\sin^2\alpha)\cos\alpha}{\sqrt{1 - n^2\sin^2\alpha}} - n\cos 2\alpha\right]Q_t d\alpha d\beta
\tag{14}
$$

Substituting (13) and (14) into (10), following is obtained

$$
\begin{aligned}
\mathbf{E}^r(x, z) &= \frac{k}{2\pi}\int_0^T\int_0^{2\pi}\mathbf{E}_T(\xi, \eta)\left[\frac{E\rho D_t Q_t}{Pq_z^e}\right]^{\frac{1}{2}} \\
&\quad \times \exp\left[-jk\left(Kr\sin\alpha\sin\theta_0\cos(\phi_0 - \beta)\right)\right] \\
&\quad \times \exp\left[-jk\left(S_0 + (\zeta_1 - \zeta)q_z - \rho K(\alpha)\sin\alpha + (z - \zeta_1)q_z^e\right)\right]d\alpha d\beta
\end{aligned}
\tag{15}
$$

Subtended angle T of lens is given by

$$
T = \arctan\left(\frac{1}{\sqrt{n - 1}}\frac{a}{\sqrt{(n + 1)a^2 + (n - 1)f^2}}\right)
$$

The initial value $\mathbf{A}_0(\xi, \eta)$ in (15) may be obtained by using GO theory. The transmitted field by plano convex into uniaxial crystal is given by [34]

$$\mathbf{E_T} = T_\alpha^{ee}\Big(T_\perp \sin^2 \beta + [T_\perp + (n\sin^2\alpha + \cos\alpha\sqrt{1-n^2\sin^2\alpha})T_\parallel \cos^2\beta\Big)\mathbf{i}_x$$

$$+ T_\alpha^{ee}\Big(-\sin\beta\cos\beta + (n\sin^2\alpha + \cos\alpha\sqrt{1-n^2\sin^2\alpha})\Big)T_\parallel]\mathbf{i}_y$$

$$+ T_\alpha^{ee}\Big(T_\parallel(n\cos\alpha - \sqrt{1-n^2\sin^2\alpha})\Big)\sin\alpha\cos\beta]\mathbf{i}_z$$

where

$$T_\parallel = \frac{2n\cos\alpha}{\cos\alpha + n\sqrt{1-n^2\sin^2\alpha}}, \qquad T_\perp = \frac{2n\cos\alpha}{n\cos\alpha + \sqrt{1-n^2\sin^2\alpha}}$$

$T_\parallel$ denotes transmission coefficients of plano convex lens, T_α^{ee} denotes transmission coefficients.

The direction of the optical axis in in the uniaxial crystal along the unit vector $\hat{s}$ given by.

$$\hat{s} = \sin\theta\cos\phi\mathbf{i}_x + \sin\theta\sin\phi\mathbf{i}_y + \cos\theta\mathbf{i}_z$$

The transmission coefficients may be obtained [5–8] by

$$T_\alpha^{ee} = \frac{2\mu q^2 q_x q_z}{\mu_1(q^o)^2 q_z A^{et} - \mu q^2 B^{et}}$$

where

$$A^{et} = \cos\theta Q_t - \sin(\theta + \phi)q_z^e$$
$$B^{et} = \sin(\theta + \phi)(q^o)^2 - \sin(\theta + \phi)Q_t^2 - q_z^e Q_t\cos\theta$$

Finally the expression which is valid around the caustic is

$$\mathbf{E}^r(x,z) = \frac{k}{2\pi}\int_0^T\int_0^{2\pi}\mathbf{E_T}\left[\frac{E\rho D_t Q_t}{Pq_z^e}\right]^{\frac{1}{2}}\exp[-jk(Kr\sin\alpha\sin\theta_0\cos(\phi_0 - \beta))]$$

$$\times \exp\left[-jk\Big(S_0 + (\zeta_1 - \zeta)q_z - \rho K(\alpha)\sin\alpha + (z - \zeta_1)q_z^e\Big)\right]d\alpha d\beta$$

$$(16)$$

The integration with respect to β can be performed by using the integral representation of Bessel function. The results are expressed as

$$E_x = \frac{k}{2}\Big[P_1(r, \theta) - Q_1(r, \theta)\cos 2\phi_0\Big] \tag{17}$$

$$E_y = \frac{k}{2}Q_1(r, \theta)\sin 2\phi_0 \tag{18}$$

$$E_z = jkR_1(r, \theta\cos\phi_0 \tag{19}$$

where

$$
\begin{aligned}
P_1(r,\theta) &= \int_0^T J_0(kK(\alpha)r\sin\theta\cos\alpha)\left[\frac{E\rho D_t Q_t}{Pq_z^e}\right]^{\frac{1}{2}} \\
&\quad \times [T_\perp + (n\sin^2\alpha + \cos\alpha\sqrt{1 - n^2\sin^2\alpha})T_\parallel] \\
&\quad \times \exp\left[-jk\left(S_0 + (\zeta_1 - \zeta)q_z - \rho K(\alpha)\sin\alpha + (z - \zeta_1)q_z^e\right)\right. \\
&\quad \left. +j\frac{\pi}{2}sign(p)\right]d\alpha \\
Q_1(r,\theta) &= \int_0^T J_1(kK(\alpha)r\sin\theta\cos\alpha)\left[\frac{E\rho D_t Q_t}{Pq_z^e}\right]^{\frac{1}{2}} \\
&\quad \times \left[\left(T_\parallel(n\cos\alpha - \sqrt{1 - n^2\sin^2\alpha})\right)\sin\alpha\right] \\
&\quad \times \exp\left[-jk\left(S_0 + (\zeta_1 - \zeta)q_z - \rho K(\alpha)\sin\alpha + (z - \zeta_1)q_z^e\right)\right. \\
&\quad \left. \times +j\frac{\pi}{2}sign(p)\right]d\alpha \\
R_1(r,\theta) &= \int_0^T J_2(kK(\alpha)r\sin\theta\cos\alpha)\left[\frac{E\rho D_t Q_t}{Pq_z^e}\right]^{\frac{1}{2}} \\
&\quad \times \left[\left(-T_\perp + (n\sin^2\alpha + \cos\alpha\sqrt{1 - n^2\sin^2\alpha})T_\parallel\right)\right. \\
&\quad \times \exp\left[-jk\left(S_0 + (\zeta_1 - \zeta)q_z - \rho K(\alpha)\sin\alpha + (z - \zeta_1)q_z^e\right)\right. \\
&\quad \left.\left. \times +j\frac{\pi}{2}sign(p)\right]d\alpha \right.
\end{aligned}
$$

where $sign(p) = 1$ for $\frac{E\rho DQ_t}{Pq_z^e} > 0$ and $sign(p) = -1$ for $\frac{E\rho DQ_t}{Pq_z^e} < 0$

4. COMPARISON TO THE HUYGENS-KIRCHHOFF PRINCIPLE

The expression based on the Huygens-Kirchhoff Principle is given by

$$
E(x,y,z) = -\frac{jk}{2\pi}\int\int_C A_0(\xi,\eta)\frac{\exp(-jkr)}{r}\left[-jk\left(S_0\right)\right]d\xi d\eta \quad (20)
$$

where

$$
r = \sqrt{(x-\xi_1)^2 + (y-\eta_1)^2 + f^2}, \quad S_0 = n\zeta + t_1, \quad A_0(\xi,\eta) = \frac{1}{\sqrt{J(t)}}
$$

5. RESULTS AND DISCUSSION

Field pattern around the caustic of a plano-convex lens into uniaxial crystal are determined by performing the integration, in equations (17) and (20), numerically by using Mathcad software. Figure 2 provides comparison between Maslov's method and Kirchhoff's approximation. The solid line shows the results obtained using Maslov's method while dashed line is for result obtained using Kirchhoff's approximation which are in good agreement. This agreement proof the validity of Maslov's method. It is difficult to determine which method provides more precise solution, but each method give a similar order of accuracy. Figure 3 show comparison between two situations, one deals with isotropic medium (solid line) and other deals with uniaxial crystal(dashed line) with optical axis making angles at $\theta = 0°$. The results are displayed in Figure 3 show that the maximum intensities are indeed the same, as expected, but the focus in the crystal is shifted towards the interface compared to the focus in the isotropic medium. The crystal can be replaced by an isotropic medium by putting $n^e = n^o = 1$.

Figure 4 shows comparison of field distribution at different orientation of optical axis that is at $\theta = 0°$, $\theta = 30°$, $\theta = 45°$ and $\theta = 60°$. Figure 5 also show comparison of field distribution at different orientation of optical axis that is at $\theta = 0°$, $\theta = 45°$, $\theta = 75°$ and $\theta = 90°$. It is shown that the focal area for a negative uniaxial crystal is displaced in the x and z directions as the angle θ is increased
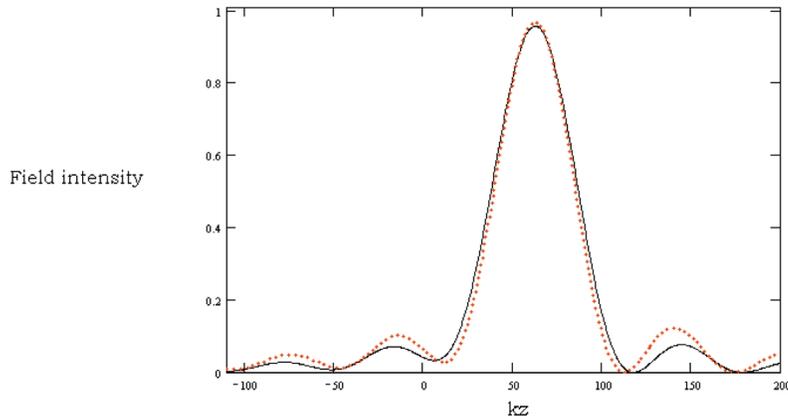
Figure 2. Comparison of normalized field distribution around focal point by Maslov's method (solid line) and Kirchhoff's principle (dashed line).
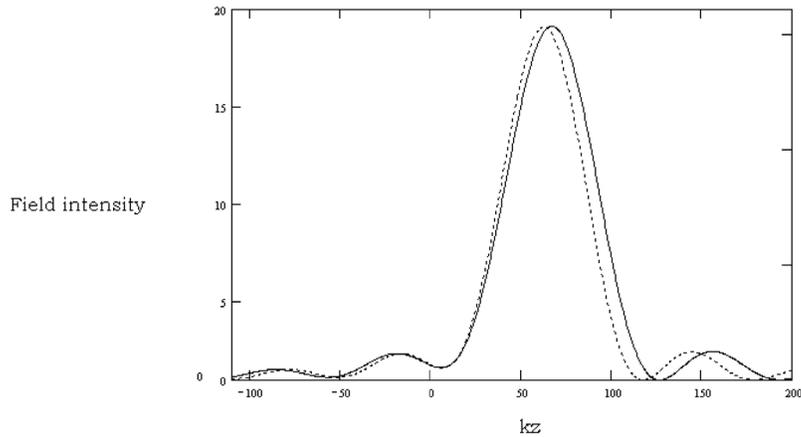
Figure 3. Comparison of field distribution around the focal point between isotropic media (solid line) and uniaxial crystal (dashed line).
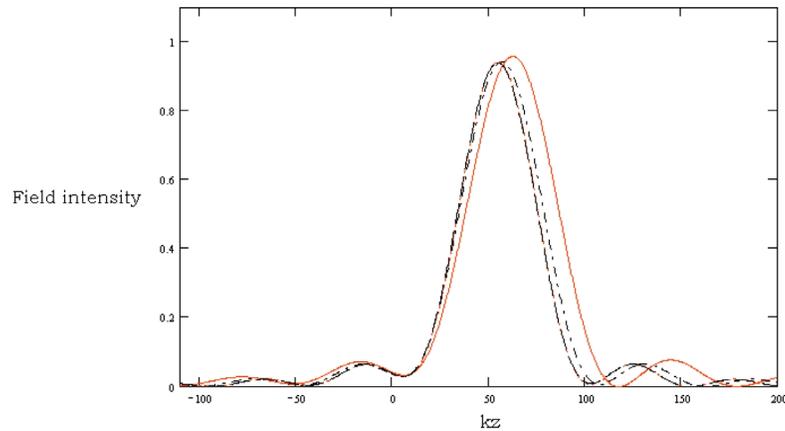
Figure 4. Comparison of field distribution around focal region at different orientations of optical axis i.e., at theta $= 0$ (solid line), theta $= 30$ (dot line), theta $= 45$ (dashed line) and theta $= 60$ (dashdot line).

from $\theta = 0°$. If we continue to increase the angle θ, we will obtain a maximum displacement of the focal area when $\theta = 45°$. If the angle θ is monotonically increased above $\theta = 45°$, then the displacement of the focal area will be monotonically reduced until θ approaches $\theta = 90°$.

Throughout the discussion, for uniaxial crystal case, we have used $LiNbO_3$, which has ordinary refractive index of $n^o = 2.208$ and
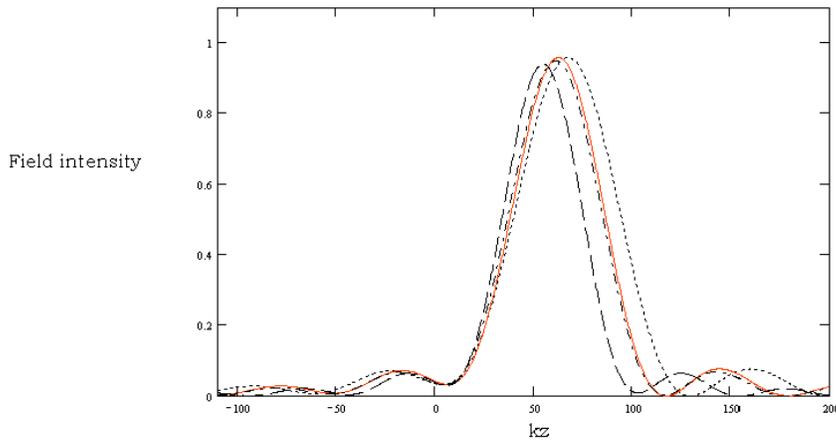
Figure 5. Comparison of field distribution around focal region at different orientations of optical axis i.e., at theta = 0 (solid line), theta = 45 (dashed line) and theta = 75 (dashdot line) and theta = 90 (dot line).
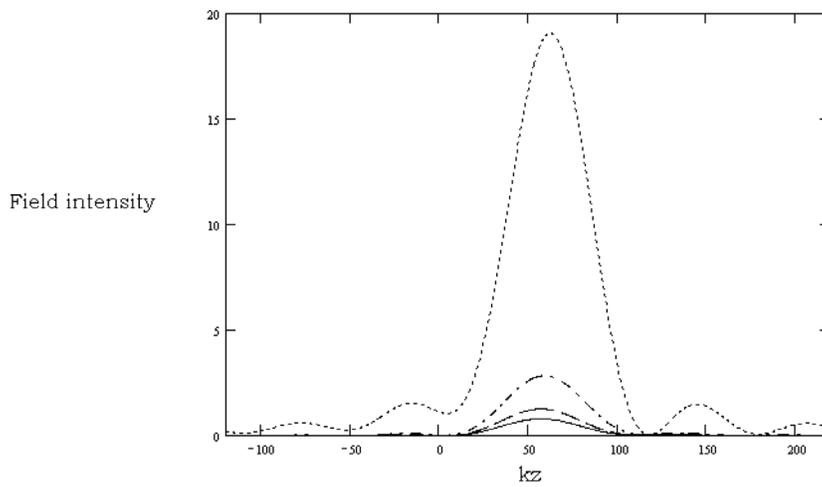
Figure 6. Field intensity variation at different orientations of optical axis i.e., at theta = 0 (dot line), theta = 15 (dashdot line), theta = 30 (dashed line) and theta = 45 (solid line).
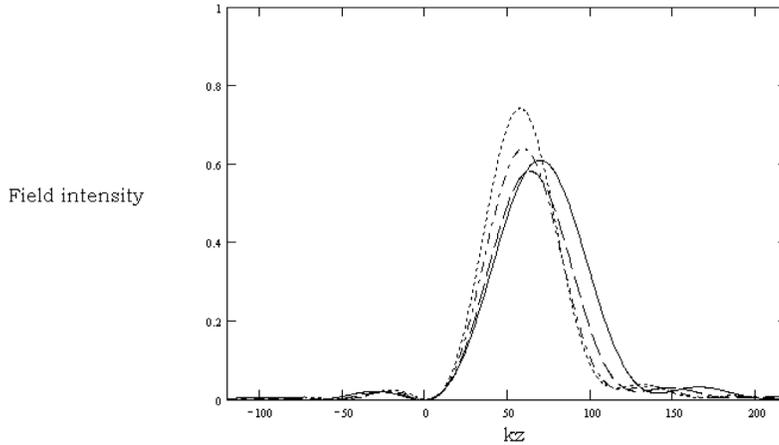
Figure 7. Field intensity variation at different orientations of optical axis i.e., at theta $= 50$ (dot line), theta $= 60$ (dashdot line), theta $= 75$ (dashed line) and theta $= 90$ (solid line).

extraordinary refractive index of $n^e = 2.300$. The distance between the rear face of the lens and front face of uniaxial crystal $kd_1 = 5$ from the plane uniaxial interface. It may be noted that, focal point in the absence of the crystal would be at a distance of $kf = 15$, refractive index of plano lens is $n = 3.84$ and radius is $ka = 15$. Figure 6 and Figure 7 show variation of field intensities at different orientation of optical axis that is at $\theta = 0°$, $\theta = 15°$, $\theta = 30°$, $\theta = 45°$, $\theta = 50°$, $\theta = 60°$, $\theta = 75°$ and $\theta = 90°$ respectively.

APPENDIX A. EVALUATION OF THE $J(t)$

$$
D(t) = \frac{\partial(x, y, z)}{\partial(\xi_1, \eta_1, t)} = \begin{vmatrix} 1 + \dfrac{\partial q_x}{\partial \xi_1}t & \dfrac{\partial q_y}{\partial \xi_1}t & \dfrac{\partial \zeta_1}{\partial \xi_1} + \dfrac{\partial q_z^e}{\partial \xi_1}t \\[2mm] \dfrac{\partial q_x}{\partial \eta_1}t & 1 + \dfrac{\partial q_y}{\partial \eta_1}t & \dfrac{\partial \zeta_1}{\partial \eta_1} + \dfrac{\partial q_z^e}{\partial \eta_1}t \\[2mm] q_x & (q_y) & q_z^e \end{vmatrix}
$$

$$
= Ut^2 + Vt + W \tag{A1}
$$

where U, V, W are

$$
U = \left(\frac{\partial(q_y)}{\partial \xi_1} \frac{\partial q_z^e}{\partial \eta_1} - \frac{\partial(q_z^e)}{\partial \xi_1} \frac{\partial(q_y)}{\partial \eta_1} \right) (q_x)
$$

$$+ \left(\frac{\partial(q_x)}{\partial \eta_1} \frac{\partial(q_z^e)}{\partial \xi_1} - \frac{\partial(q_z^e)}{\partial \eta_1} \frac{\partial(q_x)}{\partial \xi_1} \right) (q_y)$$

$$+ \left(\frac{\partial(q_x)}{\partial \xi_1} \frac{\partial(q_y)}{\partial \eta_1} - \frac{\partial(q_y)}{\partial \xi_1} \frac{\partial(q_x)}{\partial \eta_1} \right) (q_z^e)$$

$$V = \left(\frac{\partial(q_y)}{\partial \xi_1} \frac{\partial \zeta_1}{\partial \eta_1} - \frac{\partial \zeta_1}{\partial \xi_1} \frac{\partial(q_y)}{\partial \eta_1} - \frac{\partial(q_z^e)}{\partial \xi_1} \right) (q_x)$$

$$+ \left(\frac{\partial(q_x)}{\partial \eta_1} \frac{\partial \zeta_1}{\partial \xi_1} - \frac{\partial \zeta_1}{\partial \eta_1} \frac{\partial(q_x)}{\partial \xi_1} - \frac{\partial(q_z^e)}{\partial \eta_1} \right) (q_y)$$

$$+ \left(\frac{\partial(q_y)}{\partial \eta_1} + \frac{\partial(q_x)}{\partial \xi_1} \right) (q_z^e)$$

$$W = \left(\frac{\partial \zeta_1}{\partial \xi_1} (q_x) + \frac{\partial \zeta_1}{\partial \eta_1} (q_y) \right) + (q_z^e) \qquad (A2)$$

We may rewrite the values of U, V and W by using the following relations

$$\frac{\partial(q_x)}{\partial \xi_1} = \frac{\partial Q_t(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \xi_1} \cos \beta - Q_t(\alpha) \frac{\partial \beta}{\partial \xi_1} \sin \beta$$

$$= -PD \cos^2 \beta + \frac{Q_t}{\rho} \sin^2 \beta$$

$$\frac{\partial(q_y)}{\partial \xi_1} = \frac{\partial Q_t(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \xi_1} \sin \beta - Q_t(\alpha) \frac{\partial \beta}{\partial \xi_1} \cos \beta$$

$$= - \left(PD + \frac{Q_t(\alpha)}{\rho} \right) \cos \beta \sin \beta$$

$$\frac{\partial(q_x)}{\partial \eta_1} = \frac{\partial Q_t(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \eta_1} \cos \beta - Q_t(\alpha) \frac{\partial \beta}{\partial \eta_1} \sin \beta$$

$$= - \left(PD + \frac{Q_t(\alpha)}{\rho} \right) \cos \beta \sin \beta$$

$$\frac{\partial(q_y)}{\partial \eta_1} = \frac{\partial Q_t(\alpha)}{\partial \alpha} \frac{\partial \alpha}{\partial \eta_1} \sin \beta - Q_t(\alpha) \frac{\partial \beta}{\partial \eta_1} \cos \beta$$

$$= -PD \sin^2 \beta + \frac{Q_t}{\rho} \cos^2 \beta$$

$$\frac{\partial q_z^e}{\partial \xi_1} = \frac{\partial q_z^e}{\partial \alpha} \frac{\partial \alpha}{\partial \xi_1} = -P \frac{\partial q_z^e}{\partial \alpha} \cos \beta$$

$$\frac{\partial q_z^e}{\partial \eta_1} = \frac{\partial q_z^e}{\partial \alpha} \frac{\partial \alpha}{\partial \eta_1} = -P \frac{\partial q_z^e}{\partial \alpha} \sin \beta$$

where

$$\tan \alpha = -g'(\rho), \quad \tan \beta = \frac{\eta_1}{\xi_1}, \quad \frac{\partial \alpha}{\partial \xi_1} = P \cos \beta, \quad \frac{\partial \alpha}{\partial \xi_1} = P \sin \beta$$

$$\frac{\partial \beta}{\partial \xi_1} = -\frac{\sin \beta}{\rho}, \quad \frac{\partial \beta}{\partial \eta_1} = \frac{\cos \beta}{\rho}, \quad \frac{\partial \zeta_1}{\partial \xi_1} = g'(\rho)\cos \beta, \quad \frac{\partial \zeta_1}{\partial \eta_1} = g'(\rho)\sin \beta$$

The new expressions for U, V and W are given by

$$U = P\frac{Q_t(\alpha)}{\rho}(Q_t(\alpha)D_1 - q_z^e D)$$

$$V = \frac{Q_t(\alpha)}{\rho}(q_z^e + Q_t \tan \alpha) + P(Q_t(\alpha)D_1 - q_z^e D)$$

$$W = q_z^e + Q_t \tan \alpha$$

$$D_1 = \frac{\partial q_z^e}{\partial \alpha} = G\left(1 - \frac{1}{\sqrt{B}}\frac{K(\alpha)\sin \alpha}{1 + \chi \cos^2 \theta} + \frac{A}{\sqrt{B}\chi \cos^2 \theta}\right)$$

$$G = -\frac{\chi K(\alpha)\sin \theta \cos \theta}{1 + \chi \cos^2 \theta}\left(\frac{-n^2 \sin \alpha \sin 2\alpha}{2\sqrt{2 - n^2 \sin^2 \alpha}} + n^2 \sin^2 \alpha + K(\alpha)\cos \alpha\right)$$

where Hence we have

$$D(t) = Ut^2 + Vt + W = (P\frac{U_0}{E}t + 1)(\frac{Q_t(\alpha)}{\rho}t + 1)$$

where

$$P = \frac{(\sqrt{n^2 - 1}) \cdot \frac{n-1}{n+1}f^2}{[n^2\xi^2 + (n^2 - 1)^2 f^2][\xi^2 + \frac{n^2-1}{n+1}f^2]^{\frac{1}{2}}}$$

$$\rho = \frac{(n - 1)f \tan \alpha}{\sqrt{1 - (n^2 - 1)\tan^2 \alpha}}$$

$$U_0 = Q_t\frac{\partial q_z^e(\alpha)}{\partial \alpha} - q_z^e\frac{\partial Q_t(\alpha)}{\partial \alpha})$$

$$E = q_z^e + Q_t \tan \alpha$$

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